THE FOUNDATIONS OF THE CALCULUS OF VARIATIONS IN THE LARGE IN m-SPACE (FIRST PAPER)*

BY

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1. Introduction. The calculus of variations in the large as developed by the author may be roughly divided into two parts.

The first part involves the characterization of the number of conjugate points or focal points on a given extremal in terms of the type number of a basic quadratic form, and furnishes a connecting link between the theory of such extremals and the theory of critical points of functions.

The second part goes beyond the properties of one extremal alone; it involves families of extremals, families both closed and open. It necessarily leads to considerations which belong to analysis situs. It seeks to prove the existence of extremals on open or closed manifolds regardless of whether or not the extremals give minima.

The present paper consists of three parts of which the first two set up and study the fundamental forms associated respectively with an extremal joining two points, and an extremal cut transversally by a given manifold.

The third part presents a relatively complete picture of the theory in the large of extremals from a given point cut transversally by a given closed manifold. It gives strong existence theorems immediately applicable, for example, to the question of how many straight lines one can draw from a point normal to a given manifold. For the case where the manifold is homeomorphic to an $(m-1)$-sphere it can be shown that the relations found in this part are exhaustive.

The first two parts taken with the *deformation theory* previously developed by the author† will culminate in a second paper which will give the central existence theorems in the large for extremals joining two fixed points.

The first two parts taken in quite another way will lead to a generalization of the famous Sturm separation and comparison theorems for the case of a system of n linear, second-order, homogeneous, self-adjoint differential

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† Marston Morse, *The foundations of a theory in the calculus of variations in the large*, these Transactions, vol. 30 (1928), pp. 213–274.

equations. The methods and the results of this paper are also extensible to the case of two variable end points.

I. Extremals through fixed points

2. The integrand* $F(x, r)$. Let R be a closed region in the space of the variables $(x_1, \cdots, x_m) = (x)$. Let

$$F(x_1, \cdots, x_m, r_1, \cdots, r_m) = F(x,r)$$

be a function of class C''' for (x) in R and (r) any set not (0). We suppose further that F is homogeneous in that

$$F(x_1, \cdots, x_m, kr_1, \cdots, kr_m) = kF(x,r) \tag{2.1}$$

for any positive constant k. We shall start with the integral in the usual parametric form

$$J = \int_{t_0}^{t_1} F(x,\dot{x})dt \tag{2.2}$$

where $(\dot{x})$ stands for the set of derivatives of (x) with respect to t.

Let g be an extremal segment lying in R. We suppose that g is an ordinary curve of class C'. Denote the second partial derivative of $F(x, r)$ with respect to r_i and r_j by F_{ij}. Along g we assume that F is positively regular, that is, that†

$$F_{ij}(x,\dot{x})\eta_i\eta_j > 0 \qquad (i,j = 1,2, \cdots, m) \tag{2.3}$$

for $(x, \dot{x})$ taken along g, and for (η) any set not (0) nor proportional to $(\dot{x})$.

Because of the homogeneity relation (2.1) we have the m identities

$$r_iF_{ij}(x,r) \equiv 0 \qquad (j = 1,2, \cdots, m) \tag{2.4}$$

from which it appears that

$$|F_{ij}| \equiv 0. \tag{2.5}$$

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* For that which concerns the classical theory of the calculus of variations in space the reader may refer to the following articles:

Mason and Bliss, *The properties of curves in space which minimize a definite integral*, these Transactions, vol. 9 (1908), pp. 440–466.

Bliss, *The Weierstrass E-function for problems of the calculus of variations in space*, these Transactions, vol. 15 (1914), pp. 369–378.

Bliss, *Jacobi's condition for problems of the calculus of variations in parametric form*, these Transactions, vol. 17 (1916), pp. 195–206.

Hadamard, *Leçons sur le Calcul des Variations*, vol. 1, Paris, 1910.

Carathéodory, *Die Methode der geodätischen Äquidistanten und das Problem von Lagrange*, Acta Mathematica, vol. 47 (1926), pp. 199–236.

† Here and elsewhere we adopt the convention that the repetition of a subscript indicates a summation with respect to that subscript.

The generalized Weierstrassian function $F_1(x, r)$ can here be conveniently defined as the sum of the principal minors of order $m-1$ of the determinant (2.5) divided by the sum of the squares of the r_i's.*

For $(x, \dot{x})$ on g we now form the characteristic determinant $D(\lambda)$ of the determinant (2.5). We see that one of the roots of $D(\lambda)=0$ must be zero on account of (2.5). The remaining roots must be positive on account of (2.3). A consequence is that the coefficient of λ in $D(\lambda)$ must be negative. But this coefficient is

$$-(\dot{x}_1^2 + \cdots + \dot{x}_m^2)F_1(x, \dot{x})$$

so that F_1 is thereby proved to be positive along g.

3. The invariance of our hypotheses under admissible transformations. By an admissible transformation from the variables (x) to a new set of variables (z) will be understood a transformation of the form

$$x_i = \phi_i(z_1, \cdots, z_m) \qquad (i = 1, 2, \cdots, m), \tag{3.1}$$

where the functions ϕ_i are of class C''' in their arguments and where the jacobian of the ϕ_i's with respect to the z_i's is never zero. We obtain a new integrand $G(z, r)$ by setting

$$G(z_1, \cdots, z_m, r_1, \cdots, r_m) = F\left(\phi_1, \cdots, \phi_m, \frac{\partial \phi_1}{\partial z_i} r_i, \cdots, \frac{\partial \phi_m}{\partial z_i} r_i\right). \tag{3.2}$$

The positive regularity of F along g implies the positive regularity of G along the transform of g, say γ. That is, it follows from (2.3) and (3.2) that

$$G_{ij}(z, \dot{z})\zeta_i\zeta_j > 0 \qquad (i, j = 1, 2, \cdots, m) \tag{3.3}$$

for $(z, \dot{z})$ on γ, and for (ζ) any set not (0) nor proportional to $(\dot{z})$. The proof of this statement appears readily if one obtains the second partial derivatives of G in terms of those of F.

That G possesses the same homogeneity property as does F, follows from (3.2).

4. A transformation carrying g into a straight line. The following lemma will greatly simplify our future work.

Lemma. *There exists an admissible transformation T of the variables (x) into variables (z) under which the image of g is a segment of the z_1 axis.*

Suppose g is given in terms of its arc length in the form

$$x_i = h_i(s) \qquad (i = 1, 2, \cdots, m). \tag{4.1}$$

* See Bliss, second paper cited, p. 378, for a differently phrased but equivalent definition.

We now state the following preliminary proposition.

(A). Let the direction cosines of the direction tangent to g be represented by a point P_1 on a unit $(m-1)$-sphere S with center at the origin O. As P_1 moves continuously on S there can be chosen $m-1$ other points $P_2, \cdots, P_m$ on S which vary continuously with s, and are such that the points

$$O, P_1, \cdots, P_m$$

for no value of s lie on the same $(m-1)$-plane.

The proof of this proposition may be given in so many ways that it can very well be left to the reader.

Granting the truth of (A) consider the determinant b whose ith column gives the coördinates of the point P_i. We now replace the elements of the last $m-1$ columns of b by analytic approximations taken so close that the new determinant $a(s)$ is never zero. Let $a_{ij}(s)$ be the ijth element of $a(s)$. Our transformation T can now be defined as follows:

$$x_i = h_i(z_1) + a_{ik}(z_1)z_k \qquad (k = 2,3, \cdots, m; \; i = 1,2, \cdots, m) \tag{4.2}$$

where z_k is taken in the neighborhood of $z_k = 0$, and z_1 may take on the same values as s. We have for $z_k = 0$

$$\frac{D(x_1, \cdots, x_m)}{D(z_1, \cdots, z_m)} = a(z_1) \neq 0, \tag{4.3}$$

so that the lemma follows readily.

5. **The integral in non-parametric form.** According to the result of the last section there exists an admissible transformation T under which g is carried into a segment γ of the z_1 axis. We suppose that the positive sense along g corresponds to the positive sense of the z_1 axis. Suppose that under T the new integrand G is given by (3. 2). We now set

$$(z_1, \cdots, z_m) = (x, y_1, \cdots, y_n) = (x, y), \qquad n = m - 1,$$

and further set

$$\begin{aligned} f(x,y,p) = f(x,y_1, \cdots, y_n, p_1, \cdots, p_n) \\ = G(x, y_1, \cdots, y_n, 1, p_1, \cdots, p_n) \end{aligned} \tag{5.1}$$

for (x, y) neighboring γ and for any set (p).

For ordinary curves neighboring γ for which $\dot{z}_1 > 0$, our integral J now takes the non-parametric form

$$J = \int_a^b f(x, y, y')dx. \tag{5.2}$$

It will be convenient to set

$$\overset{0}{f}_{p_i p_j} = A_{ij}(x), \quad \overset{0}{f}_{p_i v_j} = B_{ij}(x), \quad \overset{0}{f}_{v_i v_j} = C_{ij}(x),$$

where the partial derivatives involved are evaluated for $(x, y, p) = (x, 0, 0)$, as indicated by the superscript zero.

We shall now reduce the regularity condition (3.3) *to the condition that*

$$A_{ij}(x)\zeta_i\zeta_j > 0 \qquad (i,j = 1, \cdots, n) \tag{5.3}$$

for every set $(\zeta) \neq (0)$.

Because G is homogeneous in (r) we have

$$G_{ij}(z,\dot{z})\dot{z}_j = 0. \tag{5.4}$$

In particular along γ we have

$$\dot{z}_2 = \cdots = \dot{z}_m = 0,$$

so that (5.4) reduces along γ to

$$G_{i1}(z,\dot{z})\dot{z}_1 = 0. \tag{5.5}$$

Since $\dot{z}_1 \neq 0$ along γ we infer from (5.5) that the elements in the first row and column of the matrix of the form (3.3) are zero along γ. With this understood the condition (3.3) taken along γ reduces with the aid of (5.1) to (5.3).

A particular consequence of (5.3) is that

$$|A_{ij}| \neq 0 \tag{5.6}$$

along γ.

6. The invariance of the order of a conjugate point. Let O be a point of g and (λ) the set of the m direction cosines of the direction of g at O. Let (λ) represent a point P on a unit $(m-1)$-sphere S with center at the origin. Let (α) be a set of parameters in an admissible* representation of S neighboring P, or, in other words, of the directions neighboring (λ). Suppose that $(\alpha) = (\alpha^0)$ corresponds to P. Let s be the arc length on the extremals issuing from O, measuring s from O in the positive senses of the extremals, and confining s to the set of values it has on g. It is well known that the extremals that issue from O with the directions determined by (α) can be represented in the form

$$x_i = x_i(s,\alpha) \qquad (i = 1,2, \cdots, m), \tag{6.1}$$

* That is, a representation in which the coördinates (x) are to be functions of class C''' of $m-1$ parameters (α) of such sort that not all of the jacobians of $m-1$ of the coördinates with respect to the parameters (α) are zero.

where the functions (6.1) and their first partial derivatives as to s are of class C''.

The jacobian of the functions (6.1), namely

$$(6.2)\qquad \Delta(s) = \frac{D(x_1, \cdots, x_m)}{D(s, \alpha_1, \cdots, \alpha_n)}, \quad (\alpha) = (\alpha^0), \quad n = m - 1,$$

is clearly zero for $s=0$. That the zero of $\Delta(s)$ at $s=0$ is isolated can also be readily proved.* We come now to an important definition.

The points on g, not O, at which $\Delta(s)$ vanishes are called the conjugate points of O, and the order of the vanishing of $\Delta(s)$ at each of these points the order of that conjugate point.

We turn next to the non-parametric integral (5.2), and suppose the J. D. E.† set up in the non-parametric form corresponding to the extremal γ. Suppose that under the transformation T the point O on g corresponds to the point $x=a$ on γ. Let $D(x)$ be an n-square determinant whose n columns give respectively n linearly independent solutions of the J. D. E. which vanish at $x=a$. Any other such determinant will be a non-vanishing constant times $D(x)$. We shall first prove the following

LEMMA. *Under the transformation T the zeros of $\Delta(s)$ on g correspond to the zeros of $D(x)$ on γ, and the orders of corresponding zeros are the same.*

By virtue of the transformation T the extremals given by (6.1) can be represented in the space of the new variables (x, y) in the form

$$(6.3)\qquad x = h(s, \alpha),$$

$$(6.4)\qquad y_i = k_i(s, \alpha) \qquad (i = 1, 2, \cdots, n = m - 1),$$

where it appears that the jacobian

$$(6.5)\qquad \Delta_1(s) = \frac{D(h, k_1, \cdots, k_n)}{D(s, \alpha_1, \cdots, \alpha_n)}, \qquad (\alpha) = (\alpha^0),$$

equals the jacobian $\Delta(s)$ divided by the non-vanishing jacobian of the transformation T. Thus $\Delta(s)$ and $\Delta_1(s)$ vanish together and to the same orders.

Now γ coincides with the x axis so that $h_s(s, \alpha^0) \neq 0$ on γ. We can thus solve (6.3) for s as a function of x and (α), substitute the result in (6.4), and so represent the extremals (6.1) in the form

* See Mason and Bliss, loc. cit., p. 447. It is easy to see that the determinant which multiplies U on p. 447 is not zero when (u, v) are a pair of parameters admissible in our sense.

† We write J.D.E. for the Jacobi differential equations.

$$y_i = w_i(x,\alpha) \qquad (i = 1,2, \cdots, n). \tag{6.6}$$

The jacobian

$$\Delta_2(x) = \frac{D(w_1, \cdots, w_n)}{D(\alpha_1, \cdots, \alpha_n)}, \qquad (\alpha) = (\alpha^0), \tag{6.7}$$

is related to the jacobian $\Delta_1(s)$ in that

$$h_s(s,\alpha^0)\Delta_2(x) \equiv \Delta_1(s), \tag{6.8}$$

where x and s correspond under (6.3). Hence $\Delta_2(x)$ and $\Delta(s)$ vanish at corresponding points and to the same orders.

Finally, as is well known, the jacobian $\Delta_2(x)$ consists of columns which satisfy the J. D. E. The elements of these columns all vanish at $x=a$, and lastly are linearly independent since $\Delta_2(x)\not\equiv 0$. Hence

$$\Delta_2(x) \equiv cD(x), \tag{6.9}$$

where c is a non-vanishing constant. The lemma follows directly.

The following statement is an easy corollary.

Under admissible space transformations conjugate points correspond to conjugate points and the orders of conjugate points are preserved.

7. A lemma on the order of a conjugate point. In terms of the determinant $D(x)$ of §6 we have the following lemma.

LEMMA. *If $D(x)$ vanishes to the rth order at $x=a$ the rank of $D(a)$ is $n-r$.*

For simplicity suppose $a=0$. Let $a_{ij}(x)$ be the general element of $D(x)$. Let $b(x)$ be a matrix whose general element is

$$b_{ij}(x) = a_{ij}(0) + a'_{ij}(0)x. \tag{7.1}$$

According to the theory of λ matrices (here $\lambda=x$) there exist n-square non-singular matrices $\mathbf{c}$ and d of constant elements such that*

$$\mathbf{c}b(x)d = \mathbf{e}(x) \tag{7.2}$$

where $\mathbf{e}(x)$ is a matrix all of whose elements are identically zero except those in the principal diagonal.

With $\mathbf{c}$ and d so determined let us transform the dependent variables (η) of the J. D. E. into variables (w) by setting

$$w_i = c_{ij}\eta_j \qquad (i,j = 1,2, \cdots, n), \tag{7.3}$$

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* See Bôcher, *Introduction to Higher Algebra*, Chapter XX.

where c_{ij} is the general element of $\mathbf{c}$. The transformed J. D. E. will have for solutions the columns of the matrix

$$(7.4) \qquad \mathbf{c}\mathbf{a}(x).$$

It will also have for solutions the columns of the matrix

$$(7.5) \qquad \mathbf{c}\mathbf{a}(x)\mathbf{d} = \mathbf{g}(x).$$

It follows from (7.1), (7.2), and (7.5), that

$$(7.6) \qquad e_{ij}(x) = g_{ij}(0) + g'_{ij}(0)x.$$

Further I say that none of the elements in the principal diagonal of $\mathbf{e}(x)$ are identically zero. For if so the corresponding column of $\mathbf{g}(x)$ would represent a solution of the transformed J. D. E. whose coördinates as well as their derivatives would all be zero at $x=0$, and so would be identically zero. Hence $|g(x)|$ would be identically zero contrary to (7.5).

Thus the elements in the principal diagonal of $\mathbf{e}(x)$ if suitably reordered must be of the form

$$a_1x, \cdots, a_kx, \quad a_{k+1}x + b_{k+1}, \cdots, a_nx + b_n$$

where k is an integer from 1 to n inclusive, and none of the constants

$$a_1, \cdots, a_k, b_{k+1}, \cdots, b_n$$

are zero.

By virtue of the relations (7.6) between $\mathbf{e}(x)$ and $\mathbf{g}(x)$ it appears then that $|g(x)|$ vanishes to the order k at $x=0$, and has the rank $n-k$ when $x=0$. From (7.5) it follows that $|g(x)|$ and $|a(x)|$ vanish to the same order at $x=0$, and have the same rank when $x=0$. Since $D(x)$ equals $|a(x)|$ it vanishes to the order k at $x=0$, and has the rank $n-k$ when $x=0$. Thus the lemma is proved.

8. **Two additional lemmas.** The preceding lemma leads to the following.

LEMMA 1. *If $x=b$ is a conjugate point of $x=a$ of the rth order there will be just r linearly independent solutions of the J. D. E. which vanish at both $x=b$ and $x=a$.*

Any solution which vanishes at a must be linearly dependent on the columns of $D(x)$, and conversely any such linear combination will vanish at a. If now we turn to b and write down the n conditions that a linear combination of the columns of $D(x)$ vanish at b we see from the preceding lemma that there are just r linearly independent combinations that satisfy the n conditions. The lemma follows directly.

We note the following consequence.

If $x=b$ is a conjugate point of $x=a$ of the rth order, then $x=a$ is a conjugate point of $x=b$ of the rth order.

For future use Lemma 1 will be generalized as follows. The preceding proofs hold for the generalization.

LEMMA 2. *Let $D(x)$ be a determinant whose n columns are n linearly independent solutions of the J. D. E. If $D(x)$ vanishes to the rth order at $x=c$ the number of linearly independent solutions dependent on the columns of $D(x)$ which vanish at $x=c$ equals r.*

9. **The fundamental quadratic form.** Our extremal g has been transformed into a segment γ of the x axis in the space of the variables

$$(x,y) = (x,y_1, \cdots , y_n).$$

We suppose γ bounded by the points $x=a$ and $x=b$ $(a<b)$.

Let us cut across the portion of γ for which $a<x<b$ with p successive n-planes t_i, perpendicular to γ, cutting γ respectively in points at which $x=x_i$. These n-planes divide γ into $p+1$ successive segments. Suppose these n-planes are placed so near together that no one of these $p+1$ segments contains a conjugate point of its initial end point. Let P_i be any point on t_i near γ. Let the points on γ at which $x=a$ and $x=b$, respectively, be denoted by A and B.

If the points

$$A,P_1, \cdots , P_p,B \tag{9.1}$$

are sufficiently near γ they can be successively joined by extremal segments neighboring γ. Let the resulting broken extremal be denoted by E. Let (u) be the set of $\mu=pn$ variables of which the first n are the coördinates (y) of P_1, the second those of P_2, and so on, until finally the last n are the coördinates (y) of P_p. The value of the integral J taken along E will be a function of the variables (u), and will be denoted by

$$J(u_1, \cdots , u_\mu) = J(u).$$

When $(u)=(0)$, E will become γ. Considerations similar to these taken up by the author* for the case $n=1$ show that $J(u)$ is of class C'' for (u) near (0). It clearly has a critical point when $(u)=(0)$. We come next to the terms of second order in $J(u)$, that is, to the fundamental form

$$_0J_{ij}u_iu_j \qquad (i,j = 1,2, \cdots , \mu), \tag{9.2}$$

* See Morse, loc. cit., pp. 217–219.

where ${}_0J_{ij}$ is the partial derivative of J with respect to u_i and u_j, and is evaluated for $(u) = (0)$, as indicated by the subscript 0.

Our first problem is to determine the rank and signature of this form.

10. The fundamental form and the J. D. E. As is conventional we set

$$\Omega(x,\eta,\eta') = A_{ij}\eta_i'\eta_j' + 2B_{ij}\eta_i'\eta_j + C_{ij}\eta_i\eta_j \qquad (i,j = 1, \cdots, n) \tag{10.1}$$

where the arguments (x, y, p) in the partial derivatives of $f(x, y, p)$ are $(x, 0, 0)$. The J. D. E. corresponding to the extremal γ are

$$\Omega_{\eta_i} - \frac{d\Omega_{\eta_i'}}{dx} = 0 \qquad (i = 1, \cdots, n). \tag{10.2}$$

It will be convenient to suppose that the points (9.1) lie not only in the space (x, y) but also in the space (x, η) of the J. D. E. A curve representing a solution of the J. D. E. will be called a *secondary extremal.* We see that successive points of (9.1) can be joined by secondary extremals. The set of these secondary extremals will form a broken secondary extremal which we may regard as *determined* by the set (u) of §9. With this understood we can prove the following

LEMMA. *The fundamental form may be represented as follows:*

$${}_0J_{ij}u_iu_j = \int_a^b \Omega(x,\eta,\eta')dx \qquad (i,j = 1, \cdots, \mu), \tag{10.3}$$

where (η) stands for the set of n functions of x which give the coördinates other than x of a point on the broken secondary extremal determined by (u).

To prove this let e be a variable neighboring $e=0$, and suppose the integral J evaluated along the broken extremal E *determined* when the set (u) is replaced by

$$(eu_1, \cdots, eu_\mu). \tag{10.4}$$

This value of J is given by the function

$$J(eu_1, \cdots, eu_\mu). \tag{10.5}$$

If now we differentiate the function (10.5) twice with respect to e, and then put $e=0$, we get the left hand member of (10.3).

If on the other hand we take the integral J along the curve E, differentiate under the integral sign twice with respect to e, and then set $e=0$, we will get the right hand member of (10.3). Here the functions η_i will be the partial derivatives with respect to e of the coördinates y_i of the point on E, and will thus represent a broken secondary extremal. If the conditions that E be the broken extremal determined by (10.4) be differentiated with respect to e it will follow that the functions (η) in (10.3) give that particular broken secondary extremal which is determined by (u).

The identity (10.3) follows at once.

11. The rank of the fundamental form. The theorem here is the following.

THEOREM 1. *If $x=b$ is a conjugate point of $x=a$ of the rth order the rank of the matrix of the form*

$$
{}_0J_{ij}u_iu_j = Q \qquad (i,j = 1,2, \cdots, \mu) \tag{11.1}
$$

is $\mu-r$. If $x=b$ is not conjugate to $x=a$ the rank is μ.

The proof depends upon the following statements, (A), (B), and (C).

(A). The rank of Q will be $\mu-r$ if the linear equations

$$
Q_{u_\alpha} = 0 \qquad (\alpha = 1,2, \cdots, \mu) \tag{11.2}
$$

have just r linearly independent solutions, and conversely.

(B). The equations (11.2) are necessary and sufficient conditions on a set (u) for such a set to determine a broken secondary extremal E' without corners at the points (9.1).

To prove (B) suppose (11.2) holds for some given set (u). Suppose u_α in (11.2) is the coördinate y_h of the point P_k in the set (9.1). We differentiate the integral in (10.3) with respect to u_α, integrate by parts, and obtain the following result, holding for the given set (u):

$$
Q_{u_\alpha} = \lim_{x=x_k^-} \Omega_{\eta_h'} - \lim_{x=x_k^+} \Omega_{\eta_h'} = 0. \tag{11.3}
$$

In the limiting process the arguments in $\Omega_{\eta_h'}$ are those on E'. If now we hold k fast and let $h=1, 2, \cdots, n$, (11.3) gives us n equations corresponding to the point $x=x_k$. It follows from (5.6) that these n equations are compatible only if E' has no corner at $x=x_k$.

Conversely equations (11.3) clearly hold for a given set (u) if the secondary extremal determined by (u) has no corners.

Thus the statement (B) is proved.

(C). A set of broken secondary extremals E' are linearly dependent or not according as the corresponding sets (u) are linearly dependent or not.

For if the coördinates (η) on q such curves E' satisfy a linear relation identically in x, then upon setting x successively equal to

$$
x_1, \cdots, x_p \tag{11.4}
$$

we see that the corresponding sets (u) satisfy the same linear relation.

Conversely if a linear combination of q of the sets (u) gives the null set (0), and if these q sets determine q broken secondary extremals E' then the same linear combination of the coördinates (η) on the q curves E' will be zero at each point at which x has the values

$$a, x_1, \cdots, x_p, b, \tag{11.5}$$

and hence be identically zero, since no point of (11.5) is conjugate to its predecessor.

Thus (C) is proved.

It follows from Lemma 1 of § 8 that $x=b$ is a conjugate point of $x=a$ of the rth order, when and only when there are just r linearly independent secondary extremals which pass through the points $x=a$ and $x=b$ on the x axis. The theorem now follows from statements (C), (B), and (A).

12. **The type number of the fundamental form.** A real, symmetric, non-singular quadratic form,

$$Q = a_{\alpha\beta}u_\alpha u_\beta \qquad (\alpha,\beta = 1,2,\cdots,\mu), \tag{12.1}$$

can be transformed by a suitable real, linear, non-singular transformation into a form

$$-z_1^2 - \cdots - z_q^2 + z_{q+1}^2 + \cdots + z_\mu^2 .$$

The integer q is called the *type number* of the given form.

LEMMA 1. *If the form Q is negative at each point of an r-plane π through the origin, excepting the origin, then the type number of Q is at least r.*

By a suitable non-singular linear transformation into variables (w) we first take the r-plane π into the r-plane

$$w_{r+1} = w_{r+2} = \cdots = w_\mu = 0. \tag{12.2}$$

After this transformation suppose Q takes the form

$$b_{\alpha\beta}w_\alpha w_\beta \qquad (\alpha,\beta = 1,\cdots,\mu). \tag{12.3}$$

On (12.2) Q will become a negative definite form

$$b_{\alpha\beta}w_\alpha w_\beta \qquad (\alpha,\beta = 1,\cdots,r). \tag{12.4}$$

Let A_m be that principal minor of the determinant of the form (12.3) which is obtained by striking out all but the first m rows and columns. Let $A_0=1$.

Note that A_r cannot be zero since the form (12.4) is definite. According to the well known theory of quadratic forms the variables $w_1, \cdots, w_r$ can be reordered so that no two consecutive members of the sequence

$$A_0, A_1, \cdots, A_r \tag{12.5}$$

are zero. So ordered, the number of changes of sign in (12.5) is r. The remaining variables of the set (w) can now be so reordered among themselves that no two consecutive numbers of the sequence

$$A_0, A_1, \cdots, A_\mu \tag{12.6}$$

are zero. So ordered, the number of changes of sign in (12.6) gives the type number of Q. Thus this type number is at least r and the lemma is proved.

LEMMA 2. *Suppose the coefficients of the form Q are continuous functions of a parameter t. If for $t=0$ the matrix $\mathbf{a}$ is of rank $\mu-r$, but is of rank μ for other values of t, then as t passes through $t=0$ the type number of Q can change in absolute value by at most r.*

According to the theory of symmetric quadratic forms the type number of Q will equal the number of λ roots of the characteristic matrix of $\mathbf{a}$ which are negative. These λ roots are continuous functions of the elements $a_{\alpha\beta}$ and hence of t. If $\mathbf{a}$ is of rank $\mu-r$ at $t=0$, just r of these roots will be zero. Hence r at most of these roots will change sign. The lemma follows directly.

LEMMA 3. *If $x=b$ is not conjugate to $x=a$, the type number of the form,*

$$Q = {}_0J_{ij}u_iu_j \qquad (i,j = 1, \cdots, \mu) \tag{12.7}$$

is at most equal to the sum of the orders of the conjugate points of $x=a$ preceding $x=b$.

Consider the set of x coördinates

$$a, x_1, \cdots, x_p, b \tag{12.8}$$

of §9. To prove the lemma we shall hold a fast but vary the remaining coördinates in (12.8), using, however, only sets (12.8) which are admissible in the sense of §9.

Suppose the first conjugate point of $x=a$ is $x=a_1$. We start our variation of the coördinates (12.8) with a choice of $b<a_1$. For this choice of b and under the condition (3.3) the x axis from $x=a$ to $x=b$ furnishes a proper minimum* for the integral J, so that the type number q will be zero.

Suppose that a_1 is a conjugate point of a of the rth order. When $b=a_1$ the rank of the matrix of Q, according to Theorem 1, will be $\mu-r$. According to Lemma 2 of this section an increase of b through a_1 will be accompanied by an increase of the type number of Q at most r.

It is clear that a set of coördinates (12.8) for which $b<a_1$ can be varied into any other admissible set with the same a and the same p. One way of

* See Bliss, loc. cit., second paper.

doing this is first to increase x_1 to its final position keeping the following coördinates in (12.8) so near x_1 as to be admissible, then while holding a and x_1 fast, to increase x_2 to its final position, keeping the following coördinates in (12.8) so near x_2 as to be admissible, and so on.

As b thereby increases through each point conjugate to a prior to b's final position, there will be an accompanying increase of the type number of Q at most equal to the order of the conjugate point thereby passed. The lemma follows at once.

13. **The theorem on the type number of Q.** By an *auxiliary curve* g will be understood a finite succession of secondary extremals joining $(a, 0)$ to $(b, 0)$, whose points of junction are all on the x axis, and not all of which are segments of the x axis. An auxiliary curve g will intersect the successive hyperplanes

$$x = x_1, \cdots, x = x_p$$

in successive points P_i which we may suppose given in (9.1). But the points (9.1) determine the set (u) of §9. A set (u) thereby determined by an auxiliary curve g will be termed an *auxiliary set* (u).

No auxiliary set (u) *is null.*

This follows from the fact that each point of the set (11.5) precedes its predecessor's conjugate point. Further a set of auxiliary curves g will be linearly dependent if and only if the corresponding sets (u) are linearly dependent.

The following theorem is fundamental.

THEOREM 2. *In case $x=b$ is not conjugate to $x=a$ the type number q of the form*

$$Q = {}_0J_{ij}u_iu_j \qquad (i,j = 1, \cdots, \mu) \tag{13.1}$$

will equal the sum of the orders of the conjugate points of a between $x=a$ and $x=b$.

Suppose the conjugate points of $x=a$ prior to $x=b$ have x coördinates

$$a_1 < a_2 < \cdots < a_s, \tag{13.2}$$

and their respective orders are

$$r_1, r_2, \cdots, r_s. \tag{13.3}$$

According to Lemma 3, §12, the type number q is such that

$$q \leqq r_1 + r_2 + \cdots + r_s. \tag{13.4}$$

We shall use Lemma 1 of §12 to prove that (13.4) must be an equality.

Corresponding to the ith conjugate point a_i, Lemma 1, §8, affirms the existence of r_i linearly independent secondary extremals,

$$h_j^{(i)} \qquad (j = 1, \cdots, r_i; \quad i = 1,2, \cdots, s), \tag{13.5}$$

which pass from $(a, 0)$ to $(a_i, 0)$. From the curves (13.5) we can now form r_i auxiliary curves,

$$g_j^{(i)}, \tag{13.6}$$

which are identical with the curves (13.5) as x varies from a to a_i, and are identical with the x axis as x varies from a_i to b.

Let

$$(u)_j^{(i)} \qquad (j = 1, \cdots, r_i; \quad i = 1,2, \cdots, s) \tag{13.7}$$

be the auxiliary sets (u) determined respectively by the curves (13.6). Our theorem will follow once we have proved the two following statements.

(A). *The* $r_1+r_2+ \cdots +r_s$ *sets* (u) *in* (13.7) *are linearly independent.*

(B). *Any linear combination* $(u)\neq 0$ *of the sets* (13.7) *will make the fundamental form* Q *negative.*

To prove (A) it will be sufficient to prove the auxiliary curves (13.6) are linearly independent.

If there were any linear relation between the curves (13.6) which in particular involved those for which $i=s$, this would entail a linear relation among the curves for which $i=s$ alone, since the remaining curves of the set (13.6) coincide with the x axis between $x=a_{s-1}$ and $x=a_s$, and the curves for which $i=s$ do not. Similarly there can be no linear relation between the curves (13.6) which involves the curves for which $i=s-1$, and so on down to the curves for which $i=1$.

Thus (A) is proved.

We come to the proof of (B). Let (u) be any linear combination of the sets (13.7) not $(u)=(0)$. Let g be the auxiliary curve obtained by taking the same linear combination of the η-coördinates of the curves of the set (13.6). Let E' be the broken secondary extremal determined by (u) as in §10.

Without loss of generality we can suppose that none of the points at which x equals $x_1, \cdots, x_p$ are conjugate to $x=a$. For if such were not the case a slight displacement of these points would move them from positions conjugate to $x=a$, and would not alter the type number of Q. With this understood we see that the corners of g cannot coincide with any of the corners of E', since the corners of g are at the conjugate points of $x=a$, and the corners of E' are not.

I say now that E' will make the integral in (10.3) negative.

For g will make that integral zero since g is composed of secondary extremals with corners on the x axis.* But there will be some component secondary extremal of E', say E'', without any corner, which joins the end points of a portion g' of g with a corner. Under the condition (5.3), E'' will give a smaller value to the integral than g' so that E' must make the integral negative. Statement (B) now follows from (10.3).

To return to the theorem we note that the set of linear combinations (u) of the sets (13.7) may be regarded as the set of points on an

$$r_1 + r_2 + \cdots + r_s$$

plane through the origin in a space of the points (u). On this plane Q is negative except for the origin. The theorem follows from Lemma 1 of §12.

We saw in §8 that if $x=b$ is not conjugate to $x=a$, then $x=a$ is not conjugate to $x=b$. By interchanging the rôles of b and a in the preceding proof we can prove the following:

The type number of the form Q of Theorem 2 is also equal to the sum of the orders of the conjugate points of $x=b$, between $x=a$ and $x=b$.

We have thus proved the following:

If the point $x=b$ on γ is not conjugate to $x=a$, then $x=a$ is not conjugate to $x=b$, and the sum of the orders of the conjugate points of $x=a$ between $x=a$ and $x=b$ equals the sum of the orders of the conjugate points of $x=b$ between $x=a$ and $x=b$.

II. The fundamental form for the case of one variable end point

14. The hypotheses and definition of the form. Let O be any point in the region R of §2. Except for O suppose that R is covered in a one-to-one manner by the field of extremals through O, and that each extremal through O passes out of R before a first conjugate point of O is reached. We assume that F is *positively regular* along each extremal through O. In R let Σ be any closed regular† $(m-1)$-manifold of class C'''. We assume that $F(x, r)$ is positive for each point (x) on Σ, and for the direction (r) of the field at that point, or in particular if O lies on Σ, for all directions r at O. This concludes our statement of hypotheses.

We shall next set up the fundamental quadratic form. Let

$$x_i = h_i(v_1, \cdots, v_n) \qquad (i = 1, 2, \cdots, m)$$

* See Bolza, *Variationsrechnung*, p. 62, first formula.

† That is, a manifold the neighborhood of each point of which can be admissibly represented. See footnote §6.

be an admissible representation of Σ in the neighborhood of a point P of Σ. At P suppose $(v)=(v^0)$. Let the value of the integral J taken along the field extremal from O to the point (v) on Σ be denoted by

$$(14.1) \qquad I(v_1, \cdots, v_n) = I(v) \qquad (n = m-1).$$

The function $I(v)$ will be of class* C'' in (v) for (v) neighboring (v^0). Suppose $I(v)$ has a critical point when $(v)=(v^0)$. The conditions that $I(v)$ have a critical point are the conditions that Σ cut the extremal g from O to P transversally. They are that

$$(14.2) \qquad I_{v_j} = F_{r_i}(x,\dot{x})\frac{\partial h_i}{\partial v_j} = 0 \quad (j = 1,2, \cdots, m-1\,; i = 1,2, \cdots, m),$$

where (x) gives the coördinates of P, and $(\dot{x})$ is the direction of the field at P.

If then Σ cuts the extremal g transversally the terms of the first order in $I(v)$ at (v^0) all vanish. We next turn to the terms of the second order in $I(v)$, and in particular are concerned with the rank and type of the fundamental quadratic form,

$$(14.3) \qquad Q = {}_0I_{ij}(v_i - v_i^0)(v_j - v_j^0) \qquad (i,j = 1,2, \cdots, n),$$

where the subscript zero indicates that the second partial derivatives I_{ij} are evaluated at $(v)=(v^0)$. This leads us to a study of focal points.

15. Focal† points and their orders. It is known that under our hypotheses‡ there exists a family of extremals neighboring g, passing through the points Q on Σ neighboring P in the senses of the extremals from O to Q, and cut transversally by Σ. They may be represented in the form

$$(15.1) \qquad x_i = w_i(s,v) \qquad (i = 1,2, \cdots, m)$$

where s is the arc length taken as positive when measured along these extremals from Σ in their positive senses, and where the functions w_i and their first partial derivatives with respect to s are of class C'' for (v) neighboring (v^0), and for such values of s as give points neighboring g. The jacobian

$$(15.2) \qquad \Delta(s) = \frac{D(w_1, \cdots, w_m)}{D(s,v_1, \cdots, v_n)}, \quad n = m-1, \quad (v) = (v^0)$$

does not vanish at $s=0$.

* This follows readily upon computing the first partial derivatives of $I(v)$ as in (14.2).

† An interesting paper on focal points has been written by M. B. White, *The dependence of focal points upon curvature for problems of the calculus of variations in space*, these Transactions, vol. 13 (1912), pp. 175–198.

‡ See Mason and Bliss, loc. cit., p. 448.

The points on g at which $\Delta(s)=0$ are called the focal points of P on Σ, and the order of vanishing of $\Delta(s)$ at each of these points the order of that point.

A necessary fact in what follows is that g cannot be tangent to Σ at P, as well as cut transversally by Σ at P. For the transversality conditions would then lead to the equation

$$F_{r_j}(x,\dot{x})\dot{x}_j = 0. \tag{15.3}$$

where (x) gives the coördinates of P, and $(\dot{x})$ the direction of g at P. But the left member of (15.3) equals F. Thus F would be zero contrary to our hypotheses on Σ.

16. **The reduction to non-parametric form.** As in §5, so here, the extremal g can be carried by an admissible transformation T into a segment γ of the x axis of a space of coördinates $(x, y_1, \cdots, y_n)$. Without loss of generality we can also suppose that T carries Σ into a manifold S that is orthogonal to γ at γ's final end point. Suppose g's end points O and P correspond to $x=a$ and $x=b$ on γ, respectively, with $a<b$. In the neighborhood of its intersection with γ, S can be represented in the form

$$x = \bar{x}(u_1, \cdots, u_n), \qquad y_i = u_i \qquad (i = 1,2, \cdots, n), \tag{16.1}$$

where $\bar{x}(u)$ is of class C''' for (u) neighboring (0).

The transformation T establishes a one-to-one correspondence between the points determined by (v) on Σ, and the points determined by (u) on S, at least for (u) neighboring (0). This correspondence will take the form

$$v_i = h_i(u_1, \cdots, u_n)$$

where the functions $h_i(u)$ are of class C''' for (u) neighboring $(u)=(0)$.

By virtue of the transformation T the family of extremals neighboring γ, cut transversally by S, can be represented in terms of the variables s and (v) of (15.1) in the form

$$x = h(s,v), \; y_i = k_i(s,v) \qquad (i = 1,2, \cdots, n). \tag{16.2}$$

As in the proof of the lemma in §6, so here, we can take x instead of s as a parameter along the extremals. We shall also replace the v_i's by their values $h_i(u)$ in term of the u_i's. The family (16.2) will then take the form

$$y_i = a^{(i)}(x,u) \qquad (i = 1,2, \cdots, n). \tag{16.3}$$

As in the proof of the lemma of §6, so here, a consideration of our successive transformations leads to the following results.

The focal points of Σ relative to the end point of g on Σ correspond under T to the zeros on γ of the jacobian

$$\frac{D(a^{(1)}, \cdots, a^{(n)})}{D(u_1, \cdots, u_n)} = \Delta_1(x), \quad (u) = (0), \tag{16.4}$$

and the orders of these focal points equal the orders of the corresponding zeros of $\Delta_1(x)$.

Under admissible space transformations focal points correspond to focal points, and their orders are preserved.

The columns of the jacobian (16.4) are solutions of the J.D.E. We shall find what conditions these columns satisfy at $x=b$.

The manifold $x=\bar{x}(u)$, $y_i=u_i$, cuts the family (16.3) transversally. We accordingly have the following identities in (u):

$$(f - p_h f_{p_h})\bar{x}_{u_i} + f_{p_i} \equiv 0 \qquad (i = 1,2, \cdots, n), \tag{16.5}$$

in which we must set

$$x = \bar{x}(u),\ y_i = u_i,\ p_h = a_x{}^{(h)}[\bar{x}(u), u] \qquad (h = 1, \cdots, n).$$

Denote the partial derivatives of $\bar{x}(u)$ with respect to u_i and u_j by $\bar{x}_{ij}$. Now let $a_{ij}(x)$ be the general element of the jacobian (16.4). If we differentiate (16.5) with respect to u_j, and set $(u)=(0)$, we find that

$$f^0\bar{x}_{ij}(0) + B_{ik}(b)a_{kj}(b) + A_{ik}(b)a'_{kj}(b) = 0 \qquad (i,j,k = 1, \cdots, n), \tag{16.6}$$

where the superscript zero means that (x, y, p) is to be replaced by $(b, 0, 0)$. Further, differentiation of the identity

$$u_i \equiv a^{(i)}[\bar{x}(u), u]$$

with respect to u_i and u_j respectively yields the result

$$a_{ij}(b) = 1, \qquad i = j,$$
$$a_{ij}(b) = 0, \qquad i \neq j.$$

Consider now any linear combination of the columns of (16.4) of the form

$$\eta_k(x) = u_j a_{kj}(x) \qquad (k,j = 1,2, \cdots, n)$$

where the u_j's are constant. Such a set takes on the values $(u_1, \cdots, u_n)$ when $x=b$. If we multiply (16.6) by u_j and sum with respect to j, we find that (η) satisfies the n relations,

$$f^0\bar{x}_{ij}(0)u_j + B_{ik}(b)\eta_k(b) + A_{ik}(b)\eta_k'(b) = 0 \qquad (i = 1, \cdots, n). \tag{16.7}$$

We conclude with the following lemma.

Lemma. *Any linear combination of the columns of* (16.4) *which takes on the values* $(u_1, \cdots, u_n)$ *when* $x=b$, *will satisfy the* n *relations* (16.7) *at* $x=b$, *and as a solution of the J.D.E. be uniquely determined by these values* (u) *at* $x=b$, *and by the relations* (16.7).

17. The rank of the fundamental form. Consider again the function $I(v)$ of §14 which gives the value of the integral along the extremals from O to Σ. In terms of the parameters (u) which represent S in the neighborhood of the end point of γ, $I(v)$ becomes a function

$$J(u_1, \cdots, u_n) = J(u).$$

The function $J(u)$ gives the value of the transformed integral along the extremals from the point $x=a$ on γ to the point on S at which $(y)=(u)$. The form

$$Q = {}_0J_{ij}u_iu_j \qquad (i,j = 1,2, \cdots, n),$$

in which the subscript 0 indicates that the partial derivatives are evaluated at $(u)=(0)$, will have the same rank and type number as the form (14.3).

We shall now prove the following lemma.

Lemma. *The fundamental form can be expressed as follows*:

$$(17.1) \qquad {}_0J_{ij}u_iu_j = f^0\bar{x}_{ij}(0)u_iu_j + \int_a^b \Omega(x,\eta,\eta')dx \qquad (i,j = 1, \cdots, n)$$

where (η) *stands for the coördinates* $[\eta_1(x), \cdots, \eta_n(x)]$ *on the secondary extremal joining* $x=a$ *on* γ *to the point* $(\eta)=(u)$ *on the* n*-plane* $x=b$.

Let the family of extremals which join the point $x=a$ on γ to the point at which $(y)=(u)$ on S, be represented in the form

$$y_i = m^{(i)}(x,u) \qquad (i = 1,2, \cdots, n),$$

where (u) lies in the neighborhood of $(u)=(0)$. As in §10, so here, we consider the function

$$(17.2) \qquad J(eu_1, \cdots, eu_n) = \int_a^{\bar{x}(eu)} f[x,m(x,eu),m_x(x,eu)]dx,$$

where e is a variable neighboring $e=0$.

If the two members of (17.2) be differentiated twice with respect to e, and then e be set equal to zero, (17.1) will result with

$$\eta_i(x) = u_jm_{u_j}{}^{(i)}(x,0).$$

That $\eta_i(b)=u_i$ follows upon differentiating the identity

$$u_i = m^{(i)}[\bar{x}(u), u]$$

with respect to u_j, and setting $(u) = (0)$.

The lemma is thereby proved.

The rank of Q is given by the following theorem.

THEOREM 3. *If $x = a$ on γ is a focal point of the rth order of S relative to the point $x = b$ on γ, then the rank of the matrix of the form Q is $n - r$. If $x = a$ is not a focal point of S the form is non-singular.*

As in the proof of Theorem 1, so here, we see that the rank of Q is $n - r$ if the n linear equations

$$Q_{u_i} = 0 \qquad (i = 1, 2, \cdots, n) \tag{17.3}$$

have just r linearly independent solutions (u) and conversely.

If we differentiate the right hand member of (17.1) with respect to u_i, and perform the usual integration by parts, we find that when (17.3) holds for a set (u), the functions (η) of (17.1) which take on at $x = b$ these values (u), satisfy (16.7). According to the lemma of §16 these functions (η) must be dependent on the columns of the jacobian of (16.4). But the functions (η) of (17.1) vanish when $x = a$.

We see that the number of linearly independent solutions (u) of the equations (17.3) equals the number of linearly independent solutions of the J.D.E. which are dependent on the columns of Δ_1, and which vanish at $x = a$. This number equals the order of vanishing of $\Delta_1(x)$ at $x = a$ and thus equals the order of the focal point $x = a$. Thus the theorem is proved.

18. **The type number of the form Q.** We shall now prove the following lemma.

LEMMA. *If $x = a$ on γ is not a focal point of S relative to the point $x = b$ on γ, the type number of the form Q is at most equal to the sum of the orders of the focal points of S on γ between $x = a$, and $x = b$.*

To prove this lemma we vary a from a value near b, $a < b$, to its given value. For $x = a$ nearer $x = b$ than any focal point of S, the form Q will be positively definite, since (5.3) holds along γ. As $x = a$ decreases through a focal point of order r there will be an accompanying increase of the type number of Q at most r. This follows from Lemma 2 §12 and Theorem 3. The present lemma follows directly.

The following theorem will now be proved.

THEOREM 4. *If $x = a$ on γ is not a focal point of S relative to $x = b$ on γ, the type number q of the form Q will equal the sum of the orders of the focal points of S between $x = a$ and $x = b$.*

The proof is similar to the proof of Theorem 2. The following differences should however be noted.

By an *auxiliary curve* $y_i = \eta_i(x)$ we shall here mean a curve which is made of two secondary extremals as follows. The first shall consist of the x axis from $x = a$ to a point $x = a_1$ between $x = a$ and $x = b$. The second shall pass from the point $x = a_1$ on the x axis to a point on S at which $(y) = (u) \neq (0)$, and shall be dependent on the columns of Δ_1 in (16.4).

If in (17.1) (η) be taken as the coördinates along an auxiliary curve, and if u_i be taken as $\eta_i(b)$, the right hand member of (17.1) is zero. This is readily seen if the integral in (17.1) be written in the form

$$\frac{1}{2}\int_a^b (\eta_i \Omega_{\eta_i'} + \eta_{i'} \Omega_{\eta_{i'}}) dx,$$

and the second sum be integrated by parts. The resulting integral vanishes. The right member of (17.1) then becomes

$$f^0 x_{ij}(0) u_i u_j + \tfrac{1}{2} u_i \Omega_{\eta_i'} [b, \eta(b), \eta'(b)]. \tag{18.1}$$

This expression may also be obtained by multiplying the left member of (16.7) by u_i, and summing with respect to i, and is accordingly zero.

It follows that any secondary extremal E that joins the end points of an auxiliary curve will make the right member of (17.1) negative, since E will give the integral in (17.1) a value that is smaller than that given by the auxiliary curve.

If $x = a_1$ is a focal point of order r there will be r linearly independent auxiliary curves with corners at $x = a_1$ on γ. We now continue as in the proof of Theorem 2, and complete the proof of the present theorem.

III. Theorems in the large on extremals through a fixed point

19. **The function $I(v)$.** We return to the manifold Σ of §14 and to the field of extremals through O. We assume *that O is not a focal point of* Σ. The locus of focal points at a finite distance from Σ will form in general a finite number of manifolds of dimensionality at most $m-1$. To take O therefore as a point which is not a focal point is to take the general point O.

The integral J taken along the extremals of the field from O to a point Q on Σ will be a function I of the point Q on Σ. The critical points of I are to be determined by expressing I in the neighborhood of each point P on Σ in terms of admissible local parameters (v). Since O is not a focal point on Σ these critical points will be non-degenerate, that is, points at which the fundamental quadratic form (14.3) will be of rank n. It follows that these critical points are isolated on Σ and therefore finite in number. Their type numbers will be given by Theorem 4 of §18, and will be called the type numbers of the corresponding extremals of the field.

We shall now prove the following theorem.

THEOREM 5. *Corresponding to a continuous variation of a point O' in the neighborhood of O, the end points of the extremals from O' to Σ which are cut transversally by Σ will vary continuously on Σ while the corresponding type numbers will be unchanged.*

In §14 we imposed conditions on the field of extremals through O. If O' be restricted to a sufficiently small neighborhood of O, the field of extremals through O' will satisfy these same conditions. Let $(x)=(a)$ be the coördinates of O', and $(x)=(a^0)$ the coördinates of O. Let (v) be a set of parameters in an admissible representation of Σ in the neighborhood of the point P. The integral taken along the extremal from O' to a point (v) on Σ will be a function $I(v, a)$ of (v) and (a).

The conditions that $I(v, a)$ have a critical point when considered as a function of (v) alone are

$$I_{v_i}(v,a) = 0 \qquad (i = 1,2, \cdots, n). \tag{19.1}$$

Suppose the extremal from O to P is cut transversally by Σ at P. Let $(v)=(v^0)$ correspond to P. The equations (19.1) will have an initial solution (v^0, a^0). The jacobian of the functions I_{v_j} with respect to the parameters (v) is

$$| I_{v_i v_j} | .$$

According to Theorem 3 of §17 this jacobian will not be zero when $(v, a) = (v^0, a^0)$, since O is not a focal point of Σ.

The equations (19.1) accordingly determine the parameters (v) as single-valued continuous functions of the point (a) at least for positions of (a) neighboring (a^0). Thus the end points of the extremals from O' to Σ which are cut transversally by Σ vary continuously on Σ as required.

That the type numbers of these extremals remain unchanged follows from the fact that the coefficients of the fundamental form (14.3) can now be regarded as continuous functions of (a).

20. **The fundamental theorem in the large.** We need the following lemma.

LEMMA. *At a point $(x)=(a)$ at which both $F(a, r)$ and $F_1(a, r)$ vanish for no direction (r), there will exist, corresponding to any regular $(m-1)$-dimensional manifold S through (a), at least one direction cut transversally by S at (a).*

Reference to the transversality conditions shows that a direction $(\dot{x})$ will be cut transversally by S at $(x)=(a)$, if it is cut transversally by the $(m-1)$-plane π tangent to S at $(x)=(a)$.

Let (λ) be any direction. The conditions of transversality show that there exists at least one $(m-1)$-plane ω through (a) which cuts (λ) transversally. Now let ω be continuously turned about (a) so as to coincide with π. That we can solve the transversality conditions for the directions (λ) in terms of the parameters which determine the variable position of ω follows upon consideration of a jacobian set up by Mason and Bliss.* Although this jacobian is not used by Mason and Bliss for exactly our purpose, still they show that it does not vanish if F and F_1 are not zero.

Thus as ω is continuously turned into the plane π the direction (λ) will continuously turn to a direction that is cut transversally by π. The lemma is thereby proved.

If O is not a focal point of Σ the critical points of the function I of §14 are non-degenerate. Let M_k be the number of critical points of type k. Let R_i be the ith connectivity of Σ. According to a theorem proved by the author the following relations hold:†

$$
\begin{aligned}
M_0 &\geqq 1 + (R_0 - 1),\\
M_0 - M_1 &\leqq 1 + (R_0 - 1) - (R_1 - 1),\\
M_0 - M_1 + M_2 &\geqq 1 + (R_0 - 1) - (R_1 - 1) + (R_2 - 1),\\
&\cdots\cdots\cdots\cdots\\
M_0 - M_1 + \cdots + (-1)^n M_n &= 1 + (R_0 - 1) - + \cdots + (-1)^n(R_n - 1),
\end{aligned}
\tag{20.1}
$$

where the inequality signs alternate until the final equality is reached.

By virtue of Theorem 4, §18, the preceding statement may be put into the following form.

THEOREM 6. *Suppose O is not a focal point of Σ nor on Σ. Of the extremal segments from O to Σ which are cut transversally by Σ at their ends P on Σ let M_k be the number upon which the sum of the orders of the focal points of P between O and P is k. Then between the numbers M_k and the connectivities R_i of Σ the relations* (20.1) *hold.*

If O is on Σ the above relations still hold if we count O as if it were an extremal of type zero.

* Loc. cit., p. 447.

† Marston Morse, *Relations between the critical points of a real function of n independent variables*, these Transactions, vol. 27 (1925), p. 392. The theorem proved in this reference admits an obvious generalization to closed manifolds such as Σ.

The theorem is already established except in the special case where O is at a point (a) on Σ. In this case it follows from the lemma of this section that there is at least one extremal E issuing from Σ at (a) which is cut transversally by Σ at (a). If we now take O in a position on E not (a) but sufficiently near (a), the extremal from O to (a) will be of type zero. According to the proof of Theorem 5, §19, the other extremals from O to Σ which are cut transversally by Σ will be changed at most slightly in position, but will be unchanged in type and number. Thus the theorem follows as stated when O is on Σ.

The following corollary furnishes a strong existence theorem.

COROLLARY. *If O is not a focal point of Σ, the number of extremals from O to Σ cut transversally by Σ on which the sum of the orders of the focal points is k is at least $R_k - 1$. The total number of extremals is at least*

$$1 + (R_0 - 1) + \cdots + (R_n - 1).$$

Even when O is a focal point of Σ one can use a limiting process to get strong existence theorems. Considerations of this sort will be presented in a later paper.

21. **Normals from a point to a manifold.** A beautiful geometric application of these results obtains if we consider the integral which gives the arc length. The hypotheses of §14 are here automatically fulfilled for any point O. It remains to consider the focal points.

Let P be an arbitrary point on Σ. Let P be taken as the origin, and the $(m-1)$-plane tangent to Σ at P be taken as the $(m-1)$-plane $x_m = 0$. After a suitable rotation of the remaining axes in the $(m-1)$-plane $x_m = 0$, Σ may be represented as follows:

$$2x_m = b_i x_i^2 + H \qquad (i = 1, 2, \cdots, m-1), \tag{21.1}$$

where b_i is a constant, and H is of the third order in $(x_1, \cdots, x_{m-1})$.

For such of the constants b_i as are not zero we set

$$R_i = \frac{1}{b_i},$$

and call the point P_i on the x_m axis at which $x_m = R_i$, the ith center of a principal normal curvature of Σ corresponding to P.

If a constant $b_i = 0$, we say that the corresponding center P_i is at infinity. We define the centers P_i for any axes obtained from our specialized set by a rotation or translation, by imagining each normal to Σ rigidly fixed to Σ, and each center P_i rigidly fixed to its normal.

To determine the focal points on the normal to Σ at P we take the parameters (v) of §14 as the set $(x_1, \cdots, x_{m-1})$, and measure s along the normals in the sense of increasing x_m. We make use of the representation

(21.1) to obtain the family of normals to Σ at points neighboring P. The jacobian $\Delta(s)$ of §15 evaluated for $(v)=(v^0)$ is here seen to have the form

$$\Delta(s) = (1 - b_1 s)(1 - b_2 s) \cdots (1 - b_{m-1} s).$$

Thus the sum of the orders of the focal points on a normal to Σ, between the foot P of that normal on Σ and a point O on the normal, equals the number of centers of principal normal curvature between P and O.

It is understood that for the purpose of counting the number of centers P_i between P and O, two centers P_i and P_j are to be counted separately if $i \neq j$, even if P_i and P_j are identical in position. In case O is on Σ it is also understood that O is to be counted as a single normal on which there are, of course, no centers P_i between O and P. Finally two normals from O to Σ which have the same direction are to be counted as distinct if they have different feet on Σ. With this understood Theorem 6 gives us the following:

THEOREM 7. *Let O be a point that is not a center of principal normal curvature of Σ. Of the normals from O to Σ let M_i be the number of those on which there are i centers of principal normal curvature of Σ between O and the feet of the normals on Σ. Then between these numbers M_i and the connectivities R_i of Σ the relations* (20.1) *hold.*

COROLLARY. *If O is not a center of principal normal curvature of Σ, the number of distinct normals from O to Σ on which there are k centers of principal normal curvature is at least $R_k - 1$. The total number of distinct normals is at least*

$$1 + (R_0 - 1) + \cdots + (R_n - 1).$$

22. **Examples.** Let Σ be a regular surface of genus p in three-space. Corresponding to a point O which is not a center of principal normal curvature of Σ we have

$$M_0 \geqq 1, \qquad M_1 \geqq 2p, \qquad M_2 \geqq 1.$$

Thus there will be at least $2p+2$ distinct normals from O to Σ. In addition we have the relation

$$M_0 - M_1 + M_2 = 2 - 2p.$$

Or again let Σ be a manifold in 4-space homeomorphic to a manifold obtained by identifying the opposite faces of an ordinary cube. If O is not a center of principal normal curvature of Σ we have

$$M_0 \geqq 1, \qquad M_1 \geqq 3, \qquad M_2 \geqq 3, \qquad M_3 \geqq 1.$$

There are thus at least eight normals from O to Σ.

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